ON THE CONVERGENCE OF THE DISCRETE ORDINATES AND FINITE VOLUME METHODS FOR THE SOLUTION OF THE RADIATIVE TRANSFER EQUATION

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INTRODUCTION

- **Thermal radiation** plays a key role in many relevant engineering problems (atmospheric physics, solar energy conversion systems, industrial heating, cooling and drying processes, fires, combustion systems).

- Radiative heat transfer in participating media is governed by an integro-differential equation known as the **radiative transfer equation (RTE)**.

- Among the numerical methods used to solve this equation, the **discrete ordinates method (DOM)** and the **finite volume method (FVM)** are probably the most widely used ones.

- In contrast with fluid flow problems governed by the Navier-Stokes equations, where only the spatial discretization is needed, the RTE requires both a **spatial** and an **angular discretization**.

- The purpose of the present work is to investigate the **convergence** of the DOM and FVM or two test problems where an analytical solution of the RTE is available.
Radiative Transfer Equation (RTE)

\[
\frac{dI_\eta}{ds} = -\kappa_\eta I_\eta + \kappa_\eta I_{b\eta} + \sigma_{s\eta} I_\eta - \sigma_{s\eta} \int \frac{4\pi}{4\pi} I_\eta (\vec{s}_i) \Phi_\eta (\vec{s}_i, \vec{s}) d\Omega_i
\]

1. Decrease due to absorption
2. Increase due to emission
3. Decrease due to scattering
4. Increase due to scattering
Boundary conditions for radiative transfer equation
(Diffusely emitting and reflecting opaque surfaces)

\[ I(\mathbf{r}_w) = \varepsilon(\mathbf{r}_w) I_b(\mathbf{r}_w) + \frac{\rho(\mathbf{r}_w)}{\pi} \int_{\mathbf{n} \cdot \mathbf{s'} < 0} I(\mathbf{r}_w, \mathbf{s'}) |\mathbf{n} \cdot \mathbf{s'}| \, d\Omega' \]

Incident radiation:

\[ G = \int_0^\infty \int_{4\pi} I_\eta \, d\Omega \, d\eta \]

Radiative heat flux vector

\[ q = \int_0^\infty \int_{4\pi} I_\eta \, \mathbf{s} \, d\Omega \, d\eta \]

The total solid angle range of $4\pi$ around a point in space is discretized into a set of solid angles, and the radiation intensity is assumed to be constant over each discrete solid angle.

Each partial differential equation describes the variation of the radiation intensity in space along the direction associated with a discrete solid angle.

Integrals over solid angles are approximated by numerical quadrature.
\[ \frac{dI(r, \vec{s})}{ds} = \vec{s} \cdot \nabla I(r, \vec{s}) = -\beta(r) I(r, \vec{s}) + \kappa(r) I_b(r) + \frac{\sigma_s(r)}{4\pi} \int_{4\pi} I(r, \vec{s}') \Phi(r, \vec{s}, \vec{s}') \, d\Omega' \]

- General equation of radiative transfer for an emitting, absorbing and scattering medium:

- Discrete ordinates equations:

\[ \vec{s}_m \cdot \nabla I(r, \vec{s}_m) = -\beta(r) I(r, \vec{s}_m) + \kappa(r) I_b(r) + \frac{\sigma_s(r)}{4\pi} \sum_{l=1}^{n} w_l \, I(r, \vec{s}_l) \Phi(r, \vec{s}_m, \vec{s}_l) \quad m = 1, 2, \ldots, n \]

where \( w_m \) is the quadrature weight associated with direction \( \vec{s}_m \):

\[ \vec{s}_m = \xi_m \vec{i} + \eta_m \vec{j} + \mu_m \vec{k} \]

- These equations are also valid on a spectral basis for a nongrey medium.
- Radiative heat flux in the medium:
  \[ q(r) = \sum_{m=1}^{n} w_m I_m(r) \hat{s}_m \]

- Incident radiation:
  \[ G(r) = \sum_{m=1}^{n} w_m I_m(r) \]

- Net radiative heat flux at a surface:
  \[ q \cdot \vec{n}(r_w) = \varepsilon(r_w) \left[ \pi I_b(r_w) - \sum_{\hat{n} \cdot \hat{s}_m < 0} w_m I_m(r_w) |\vec{n}_w \cdot \hat{s}_m| \right] \]

- Divergence of radiative heat flux (source term of energy equation):
  \[ \nabla \cdot q = \kappa \left( 4 \pi I_b - \sum_{m=1}^{n} w_m I_m \right) \]
\[
\int_V \tilde{s}_m \cdot \nabla I(\tilde{r}, \tilde{s}_m) \, dV = \int_V \nabla \cdot (\tilde{s}_m \, I(\tilde{r}, \tilde{s}_m)) \, dV = \\
= \int_A I(\tilde{r}, \tilde{s}_m) \, \tilde{s}_m \cdot \vec{n} \, dA = \sum_k I_{k,m} \left( \tilde{s}_m \cdot \vec{n}_k \right) A_k
\]

\[
\int_V \left( \kappa(\tilde{r}) I_b(\tilde{r}) - \beta(\tilde{r}) I(\tilde{r}, \tilde{s}_m) \right) \, dV = \left( \kappa_p I_{b,p} - \beta_p I_{p}^m \right) V
\]

\[
\int_V \frac{\sigma_s(\tilde{r})}{4\pi} \int_{4\pi} I(\tilde{r}, \tilde{s}') \Phi(\tilde{s}, \tilde{s}') d\Omega' = \frac{\sigma_{s,P}}{4\pi} V \sum_{l=1}^n I^l_P \Phi(\tilde{s}_m, \tilde{s}_l) w_l
\]

\[
\sum_k I_{k,m} \left( \tilde{s}_m \cdot \vec{n}_k \right) A_k = \left( \kappa_p I_{b,p} - \beta_p I_{p}^m \right) V + \frac{\sigma_{s,P}}{4\pi} V \sum_{l=1}^n I^l_P \Phi(\tilde{s}_m, \tilde{s}_l) w_l
\]
TEST CASE 1

- Two-dimensional square enclosure with black walls
- Emitting-absorbing medium, unity emissive power
- All walls are black and cold
- 25×25 uniform grid, $S_8$ quadrature
- Solution accuracy evaluated using the $L_1$ norm of the relative error of the incident radiation
- The error is determined with reference to the analytical solution of the discrete ordinates equations rather than the solution of the RTE. This allows the actual evaluation of the spatial discretization error, regardless of the angular discretization error
TEST CASE 1

\[ \| E_G \|_1 = \frac{1}{N} \sum_{i=1}^{N} \frac{|G_i - G_{i,\text{ref}}|}{G_{i,\text{ref}}} \]

Average relative error of the incident radiation
• Two-dimensional square enclosure with black walls
• Emitting-absorbing medium, unity emissive power
• All walls are black; top wall is hot, others are cold
• 25×25 uniform grid, $S_8$ quadrature
• Solution accuracy evaluated using the $L_1$ norm of the relative error of the incident radiation
• The error is determined with reference to the analytical solution of the discrete ordinates equations rather than the solution of the RTE. This allows the actual evaluation of the spatial discretization error, regardless of the angular discretization error
Average relative error of the incident radiation
Incident heat flux on bottom boundary

False scattering (also referred to as false diffusion or numerical smearing) is an error that arises from the spatial discretization when the direction of propagation of radiation makes an angle with the grid lines.

- False scattering may be reduced by using a finer grid and/or high order spatial discretization schemes.

Ray effect is an error that arises from the angular discretization, which approximates a continuously varying function (the radiation intensity) by a stepwise one, i.e., the radiation intensity field at a given point in space is represented by the radiation intensities at a finite number of directions.

- The ray effect is independent of the spatial discretization, i.e., it remains even if no spatial discretization error is present.
- The ray effect may be reduced by using a finer angular discretization.
False scattering (false diffusion, numerical scattering, numerical smearing) errors tend to smooth the radiation intensity field.

- Ray effect errors tend to enhance discontinuities or gradients of the radiation intensity field.

- False scattering and ray effects tend to compensate each other.
Grid refinement and/or more accurate spatial discretization schemes reduce false scattering… … but have no influence on ray effects ⇒ compensation effect disappears

Angular refinement reduces ray effects… … but has no influence on false scattering ⇒ compensation effect disappears

In both cases, solution error of DOM is smaller compared to exact solution of DOE… but may be larger compared to exact solution of RTE

Both grid refinement (or more accurate spatial discretization schemes) and angular refinement need to carried out simultaneously. Alternatively, modify the standard DOM.
CONCLUSIONS

- The solution of the RTE for grey media using the DOM (or FVM) requires both spatial and angular discretization.

- Spatial and angular discretization errors tend to compensate each other.

- If only spatial refinement is carried out, the numerical solution approaches the analytical solution of the discrete equations, but does not converge to the analytical solution of the RTE. This means that the solution accuracy is not improved and may even get worst with spatial grid refinement.

- Both spatial and angular refinements are needed to ensure that the numerical solution converges to the analytical solution of the RTE.

- If the temperature of the boundary is discontinuous, the radiation intensity field is also discontinuous, and the order of accuracy is lower than the formal order of accuracy of the discretization schemes.