A Study of Bayesian Inference based Model Extrapolation Method

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Model Interpolation and Extrapolation

Model validation is the process of determining the degree to which a model is an accurate representation of the real world from the perspective of the intended usages. In engineering applications, model validation information needs to be utilized in design decision making.

Model Interpolation is a method of evaluating new data points within the range of a discrete set of known data points.

Model Extrapolation is a method of estimating the model predictive capability of a new data point which is beyond the range of given data.
Model Interpolation and Extrapolation

- Model interpolation and extrapolation are very useful in the design and analysis of complex engineering systems

Outstanding Issues
- Existing validation methods are restricted at a particular design, model is often not validated for the entire design space
- Few or no information is provided on the model’s predictive capability to an untested region by interpolation & extrapolation

Related Studies
- Stochastic assessment, Bayes networks and confidence extrapolation, Predictive model validation, Model validation from the perspective of engineering design

Call for the development of an effective and systematic method to address the needs of model interpolation and extrapolation
Flowchart of Bayesian based Model Interpolation and Extrapolation

Start

Design DOE matrix in the design space

Perform repeated tests

Run CAE simulations

Obtain the evidence of the bias

Calculate posterior distribution of hyper-parameters of prediction bias

Build RSMs for prediction bias distribution in design space

Calculate prediction bias at new design

Calculate Prediction Intervals (PIs)

Conduct CAE simulations

Conduct new tests

Obtain the prior information of the bias

Improve DOE matrix

Acceptable?

No

Yes

Stop

Acceptable?
Bayesian Updating of the Prediction Bias

- The relationship between the test data and the CAE results

\[ Y_{\text{test}}(x) = Y_{\text{CAE}}(x) + \delta(x) + \varepsilon(x) \]

- The "reality" or "true value" of output

\[ Y_{\text{true}}(x) = Y_{\text{test}}(x) - \varepsilon(x) = Y_{\text{CAE}}(x) + \delta(x) \]

- Assume Prediction bias \( \tau(x) = \delta(x) + \varepsilon(x) \) follows \( \tau(x) \sim N(\tau_\mu(x), \tau_{\sigma^2}(x)) \)

Given the known variance \( \tau_{\sigma^2} \), the mean value \( \tau_\mu \) needs to be inferred

- Prior

\[ \tau_{\mu0} \sim N(\mu_0, \sigma_0^2) \]

\[ p(\theta \mid x) = \frac{p(x \mid \theta)p(\theta)}{\int p(x \mid \theta)p(\theta)d\theta} \]

- Posterior

\[ \tau_{\mu1} \sim N(\mu_1, \sigma_1^2) \]

\[ \mu_1 = \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{n} \varepsilon_i}{\sigma_1^2} \right) \left( \frac{1}{\sigma_0^2} + \frac{n}{\sigma_1^2} \right)^{-1} \]

\[ \sigma_1^2 = \left( \frac{1}{\sigma_0^2} + \frac{n}{\sigma_1^2} \right)^{-1} \]
Prediction Bias RSMs and Prediction Interval for New Design

- Two Response Surface Models (RSMs) are built for the **mean** and **standard deviation** of the Prediction Bias.

- The probability $p$ that the test data will fall in a given interval

$$\Pr(Y_{CAE}(x_a) + \tau_{\mu}(x_a) - b \cdot \tau_{\sigma}(x_a) \leq Y_{test}(x_a) \leq Y_{CAE}(x_a) + \tau_{\mu}(x_a) + b \cdot \tau_{\sigma}(x_a)) = p$$

where $b$ is the $100((1 + p)/2)$th percentile.

- In case $p = 95\%$, $b = 1.96$. Hence the output or the physical test result at a new design point is predicted as

$$Y_{test}(x_a) \in [Y_{CAE}(x_a) + \tau_{\mu}(x_a) - 1.96 \cdot \tau_{\sigma}(x_a), Y_{CAE}(x_a) + \tau_{\mu}(x_a) + 1.96 \cdot \tau_{\sigma}(x_a)]$$
Case Study: Problem Description

2001 Ford Taurus model from National Crash Analysis Center (NCAC) for Frontal Impact

Fig. 4 Design variable selection for main front-end structure
Case Study: Interpolation and Extrapolation

- 80 uniform DOEs
- 3 repeated tests and 1 CAE
- 65 DOE samples are used to construct the Kriging RSMs
- 15 DOE samples to validate the interpolation capability
- 25 new designs for extrapolation study

Key Performance Output Responses:

- Chest G, Crush distance

Input Variables:

- X1
- X2

Diagram showing points for interpolation, extrapolation, and validation domain.
Case Study: Kriging RSMs of Prediction Bias

- Chest G bias mean $\tau_\mu(x_a)$
- Crush Distance bias mean $\tau_\mu(x_a)$
- Chest G bias Standard Deviation $\tau_\sigma(x_a)$
- Crush Distance bias Standard Deviation $\tau_\sigma(x_a)$
Case Study: Interpolation

The prediction intervals with 95% confidence of Chest G cover all 15 designs, and 14 out of 15 crush distances tests are within the prediction intervals.

Chest G

Crush Distance

[Graphs showing prediction intervals and successful/failed tests for Chest G and Crush Distance]
Case Study: Extrapolation

20 out of 25 extrapolation designs of Chest G and crush distance confirmation tests are within the corresponding prediction intervals with 95% confidence.

### Chest G

<table>
<thead>
<tr>
<th>Normalized Chest G Prediction of Kriging RSM</th>
<th>Prediction Interval</th>
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</thead>
<tbody>
<tr>
<td>Successful test</td>
<td>Failed test</td>
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<table>
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<tr>
<th>Normalized Crush Distance Prediction of Kriging RSM</th>
<th>Prediction Interval</th>
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<tr>
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<td>Failed test</td>
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![Normalized Chest G Prediction of Kriging RSM](chart1.png)

![Normalized Crush Distance Prediction of Kriging RSM](chart2.png)
Case Study: Extrapolation

Design #21

- Chest G

Normalized Chest G Prediction of Kriging RSM

Crush Distance

Prediction Interval

Successful test

Failed test

Normalized Crash Distance

Normalized Chest G
Summary

- This study proposes an integrated model interpolation and extrapolation framework to improve the mode predictive capability based on Bayesian inference and Response surface model.

- It provides a quantitative approach to facilitate rational decisions when only limited or no test data are available in new or extrapolated design.

- It is successfully demonstrated through a real-world example.

- Further researches will be on improving the predictive capability in extrapolation.
Thank you!